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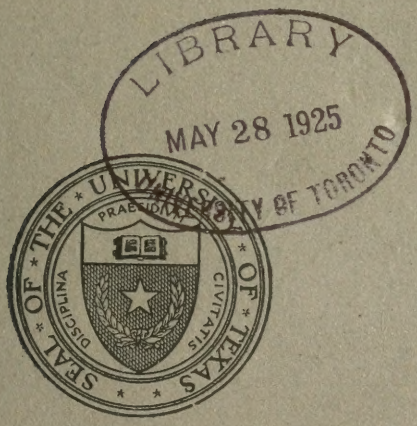
Volume X, No. 1



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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of democracy. . . . It is the only dictator that freemen acknowledge and the only security that freemen desire.

Mirabeau B. Lamar

University of Texas Bulletin
No. 2506: February 8, 1925

The Texas Mathematics Teachers' Bulletin

Volume X, No. 1

Edited by
ALBERT A. BENNETT
Associate Professor in Pure Mathematics,
and
C. M. CLEVELAND
Instructor in Applied Mathematics

This Bulletin is open to the teachers of mathematics in Texas for the expression of their views. The editors assume no responsibility for statements of facts or opinions in articles not written by them.

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INDETERMINATE EQUATIONS

By P. H. Underwood, Ball High School, Galveston

A linear equation in two unknowns, as for example,

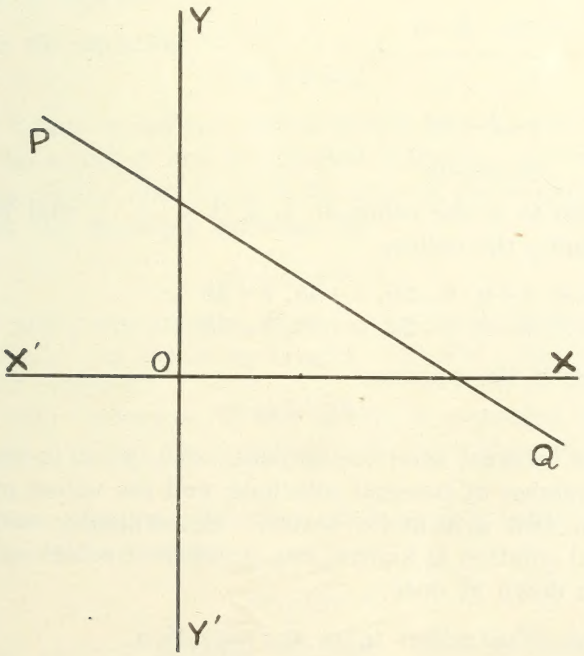
$$ax+by=c \qquad (1)$$

a, b and c any numbers positive or negative, a not zero and b not zero has an infinite number of solutions. For solving for y we get

$$y=\frac{c}{b}-\frac{a}{b}x \qquad (2)$$

Giving to x any value from $-\infty$ to $+\infty$ and substituting in (2) we have a corresponding value for y.

Graphical consideration makes this proposition evident.



If PQ is the graph of the equation

$$ax+by=c$$

the co-ordinates of every point of PQ satisfy the equation and as the number of points of PQ is infinite, the number of solutions is infinite.

Suppose that $x=h$ and $y=k$ is a solution of equation (1), then

$$ah+bk=c \qquad (4)$$

Subtract (4) from (1), then

$$a(x-h)+b(y-k)=0$$

$$\therefore\ a(x-h)=b(k-y)$$

$$\therefore\ \frac{x-h}{b}=\frac{k-y}{a}$$

Let $$\frac{x-h}{b}=\frac{k-y}{a}=n$$

$$\therefore\ x=h+bn$$

$$y=k-an$$

Assign to n the values 0, 1, 2, 3, 4 . . . . and we have for x and y the values

$$x=h,\ h+b,\ h+2b,\ h+3b,\ h+4b\ \ldots\ .$$
$$y=k,\ k-a,\ k-2a,\ k-3a,\ k-4b\ \ldots\ .$$

Hence if the equation

$$ax+by=c$$

has one integral solution, namely, $x=h$, $y=k$, it has an infinite number of integral solutions, and the values of x and y form two arithmetic series. Consequently when one integral solution is known, any number of solutions can be written down at once.

As an illustration takes the equation

$$3x+5y=78.$$

By trial we find $x=1$, $y=15$ is a solution. For other values of x add 5 and for the corresponding values of y subtract 3, the coefficients of y and x, respectively, and we have the solutions

$$x= 1,\ 6,\ 11,\ 16,\ 21,\ 26,\ 31\ \ 36$$
$$y=15,\ 12,\ 9,\ 6,\ 3,\ 0,\ -3\ -6$$

Of these, six are positive integral solutions.

As a second illustration take the equation

$$4x-7y=3.$$

By trial we find $x=-1$, $y=-1$ is a solution. By adding 7 and 4 to these values, respectively, we have the solutions.

$$x=-1,\ 6,\ 13,\ 20,\ 27,\ 34,\ 41.$$
$$y=-1,\ 3,\ 7,\ 11,\ 15,\ 19,\ 23.$$

Obviously in this case the number of positive, integral solutions is infinite.

If in the equation

$$ax+by=c$$

a and b have a common factor which is not a factor of c, then the equation has no integral solution. Let m be the common factor and divide both sides of the equation by it and let the resulting equation be

$$px+qy=c/m$$

If x and y are integers, then $px+qy$ is an integer, but by hypothesis c/m is not an integer. Hence x and y cannot be both integral.

In what follows a, b, and c will be regarded as whole numbers, positive or negative, and a and b prime to each other.

Positive Integral Solution: If a and b prime to each negative, the equation

$$ax+by=c$$

has no positive integral solution. This is obvious. If

a, b, c are all three positive, there may be no positive integral solution or a limited number of positive integral solutions. For example

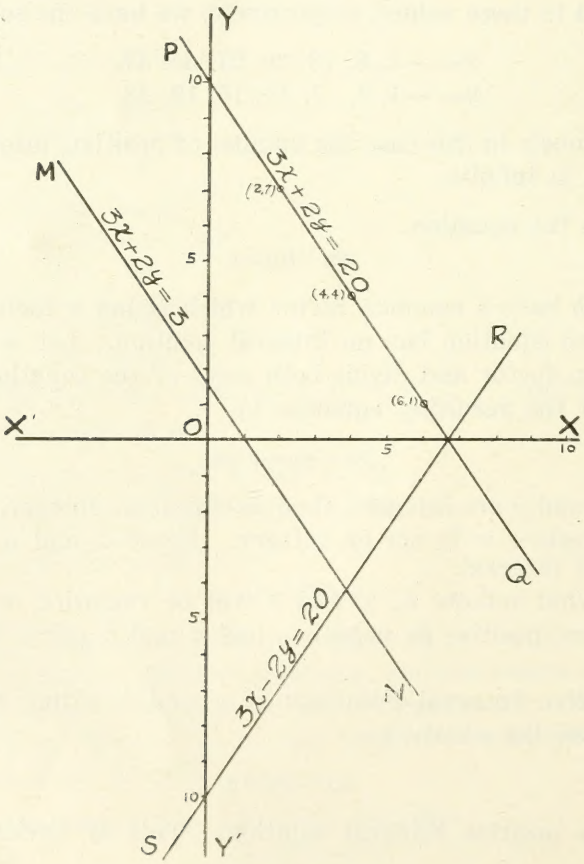
$$3x+2y=3 \tag{5}$$

has no positive integral solution.

$$3x+2y=20 \tag{6}$$

has four positive integral solutions, namely,

$$x=\ 0,\ 2,\ 4,\ 6.$$
$$y=10,\ 7,\ 4,\ 1.$$



If a and b have opposite signs there is an unlimited number of positive integral solutions. For example, the equation,

$$3x-2y=20 \qquad (7)$$

has the solutions

$$x=8,\ 10,\ 12,\ 14,\ 16,\ \ldots\ .\ 8-2(n-1) \qquad n=\infty$$
$$y=2,\ \ 5,\ \ 8,\ 11,\ 14,\ \ldots\ .\ 8-3(n-1) \qquad n=\infty$$

The graph of $3x-2y=3$ is the straight line MN. This line intersects the X-axis at the point (1, 0) and the Y-axis at the point (0, 1.5). The coördinates of the segment MN between the points (1, 0) and (0, 1.5) are positive and less than 1. The ray starting at the point (1,0) and passing through the point N has its X-coördinates positive and its Y-coördinates negative. The ray starting at the point (0, 1.5) has its X-coördinates negative and its Y-coördinates positive.

The line PQ the graph of

$$3x-2y=20$$

intersects the X-axis at the point ($6\frac{2}{3}$, 0) and the Y-axis at the point (0, 10). Between these two points are three points of the segment joining them having positive integral coördinates. For all other points of PQ one coördinate is positive and the other negative.

The line SR, the graph of

$$3x-2y=20$$

intersects the X-axis at the point ($6\frac{2}{3}$, 0) and the Y-axis at at the point (0, 10). The ray starting with the point ($6\frac{2}{3}$, 0) and passing through R consists of points whose coördinates are positive. The segment between ($6\frac{2}{3}$, 0) and (0, 10) consists of points whose X-coördinate is positive and Y-coördinate negative. The coördinates of the ray starting with (0—10) and passing through S are negative.

To determine by trial one integral solution of a linear

equation in two unknowns may be a laborious task. It is always possible to find two integers m and n in such that

$$ma+nb=1$$

when a and b are prime to each other. Indeed if m/n is the last but one convergent to the fraction a/b then

$$mb-na = \pm 1$$

This fact leads at once to the general solution of our problem.

As an illustration, consider the equation

$$17x-23y=7$$

To solve this for integers find first the convergents to the fraction 17/23.

$$\begin{array}{r} 17)23(1 \\ 17 \\ \hline 6)17(2 \\ 12 \\ \hline 5)6(1 \\ 5 \\ \hline 1)5(5 \\ 5 \\ \hline 0 \end{array}$$

The partial quotients are 1, 2, 1, 1, 5. Write these in a horizontal line.

$$\begin{array}{ccccccc} 1, & 2, & 1, & 5 & & & \\ 0 & 1 & 2 & 3 & 17 & & \\ \hline 1 & 1 & 3 & 4 & 23 & & \end{array}$$

Under the first write 0 1; under the second partial quotient, namely, 2 write 1 1, which is the first convergent, for $\frac{17}{23} = \frac{1}{1}$. Under the third partial quotient write $\frac{2}{3}$ obtained by multiplying the numerator of 1/1 by 2 and adding the previous numerator 0; and by multiplying the

denominator of 1 1 by 2 and adding the previous denominator. The next convergent $\frac{3}{4} = \frac{1\times2+1}{1\times3+1}$. The last $\frac{17}{23} = \frac{5\times3+2}{5\times4+3}$

$$\text{Now } 4\times17 - 3\times13 = -1$$
$$\text{Multiply by } 7 \quad 28\times17 - 21\times23 = -7 \qquad (8)$$
$$17x - 23y = 7 \qquad (9)$$

Adding (8) and (9) $17(x+28) - 23(y+21) = 0$

$$\therefore\ 17(x+28) = 23(y+21)$$
$$\therefore\ \frac{x+28}{23} = \frac{y+21}{17}$$

Let each fraction equal to n

$$\frac{x+28}{23} = \frac{y+21}{17} = n$$
$$x = 23n - 28$$
$$y = 17n - 21$$

For positive integral values take $n=2$

$$x = 18,\ y = 13.$$

This is the least positive integral solution.

Example 2: Solve for positive integral values of x and y.

$$29x + 38y = 3008$$

First find the convergents to $\frac{29}{38}$. The partial quotients are 1, 3, 4, 2.

```
29)38(1
   29
   --
    9)29(3
      27
      --
       2)9(4
         8
         -
          1)2(2
            2
            -
```

(2) Write these in a horizontal line

$$\begin{array}{ccccc} 1 & 3 & 4 & 2 & \\ 0 & 1 & 3 & 13 & 29 \\ \hline 1 & 1 & 4 & 17 & 38 \end{array}$$

(3) Under the first write 0 1, under the second 1/1 the first convergent under the third $\frac{3\times1+0}{3\times1+1}$ i.e. $\frac{3}{4}$, under the fourth write $\frac{4\times3+1}{4\times4+1}$ i.e. $\frac{13}{17}$ and lastly write $\frac{2\times13+3}{2\times17+4}$ i.e. $\frac{29}{38}$

$$29\times17-38\times13=-1$$
$$29\times17\times3008-38\times13\times3008=-3008$$
$$29\times51136-38\times39104=-3008$$
$$29x+38y=3008$$
$$29(x+51136)+38(y-39104)=0$$
$$29(x+51136)=38(39104)-y)$$
$$\frac{x+51136}{38}=\frac{39104-y}{29}=n$$
$$x=38n-51136$$
$$y=29104-29n$$

This is the general solution. For positive integral values of x $n>\frac{51136}{38}$ i.e. $n>1345^{13}/_{19}$

and for positive integral values of y

$n<\frac{39104}{29}$ i.e. $n<1348^{12}/_{29}$

The integral values of n between $1345^{13}/_{19}$ and $1348^{12}/_{29}$ i.e., $n=1346, 1347, 1348$ give the positive integral solutions of the equation. Assigning these values, we have

$$x=12, 50, 88$$
$$y=70, 41, 12$$

Ex. 3. Solve for positive integral values of x and y

$$26x-11y=164$$

(1) Find the convergents to 11/26.

The partial quotients are

2, 2, 1, 3

The convergents are,

$$\frac{0}{1}\quad\frac{1}{2}\quad\frac{2}{5}\quad\frac{3}{7}\quad\frac{11}{26}$$

$$\begin{array}{r|l} 11 & 26\,(2 \\ & 22 \\ \hline 4 & 11\,(2 \\ & 8 \\ \hline 3 & 4\,(1 \\ & 3 \\ \hline 1 & 3\,(3 \\ & 3 \\ \hline & 0 \end{array}$$

$$26\times3-11\times7=1$$

Multiply by 164, $\quad 26\times492-11\times1148=164$

$$26x\times11y=164$$

$$\therefore\ 26(x-492)+11(y+1148)=0$$

$$\therefore\ \frac{x-492}{-11}=\frac{y+1148}{26}$$

Let each fraction equal n.

$$\frac{x-492}{-11}=\frac{y-1148}{26}=n$$

$$\therefore\ x=492-11n$$

$$y=26n-1148$$

For positive values of x

$$n<\frac{492}{11}\ \text{i.e.}\ n<44^{8}/_{11}$$

For positive value of y

$$n>\frac{1148}{26}\ \text{i.e.}\ n>44^{2}/_{13}$$

As there is no integer intermediate in value between $44^{2}/_{13}$ and $44^{8}/_{11}$, there is no positive integral solution of the equation.

INTUITION, MOTIVATION, REASONABLENESS

By C. A. Rupp, University of Texas

About how much does a pound of tobacco cost? I hope you stop to think of a rational answer before you read my guess of a couple of dollars. At the same time I made that guess, I somehow became aware a number of qualifications, such as considering ordinary material, and estimating that the cost might be as low as a dollar, and possibly as high as four. The primary point of the initial query is to get you to consider the idea that the common run of events develop in us a feeling, an intuition, call it horse sense if you will, that enables us to give fairly reasonable answers to extraordinary question.

Is it possible for the average student to so develop his mathematical intuition that the statement of the problem arouses some scheme of attack whose result seems to him to be fairly reasonable? Naturally this question leads us so close to the frontier of our knowledge in psychology that we are in some danger of finding ideas so novel and so vague that we cannot understand them; but there is little sport in life for the timid. It would be well first to find out how much mathematical intuition the student has before we discuss its development.

One of my teachers was fond of saying that the only thing in mathematics more instructive than a contradiction was a paradox. As a bone of contention, therefore, I propose the statement that many dislike formal geometry because their intuition is already too well developed! I mean that most of the straight book theorems, and not a few of the originals have such a suspicious air of rectitude about them that it would be harder to persuade the class of their falsity than of their truth. Just try the experiment of asking your pupils to concoct a false theorem that shall be reasonably simple and yet not patently absurd. (Incidentally the author would be very glad to add to his store of good pseudo-theorems.) Granted, then, that many

of the class have already developed such a feeling for geometry that proofs of what seem quite obvious theorems (obvious in this queer, dim, intuitional way) are regarded as attempts at gilding lilies, it would appear that a strange duty of the conscientious teacher is to inspire a healthy distrust of intuition.

A good way to do this is to show the class the Möbius unilateral surface. Take a rectange *ABCD* about 10"x1", and make a half-twisted cylindrical ring by placing *C* on *A* and *D* on *B*, pasting or fastening the short ends of the rectangle together. Then try to cover one side of this ring with red crayon and the other with green, and see what happens. In *Ball's Mathematical Recreations* you can find other unexpected properties of this surface. I might suggest cutting the ring down its central line.

If the class is one convinced that their intuition may badly fool them, there is some slight danger that they may try to dispense with it completely, and try to go it blind. To keep them traveling safely down the middle path is no slight task, as we all know. It seems to me that a class ought early to get ithe idea that the intuitions very often give us the clue to the solution and some information about the rationality of the result, and for these reasons are well worthy of cultivation; but also it should be impressed hard and often that the very vagueness of the intuitional process gives it an element of untrustworthiness that must always be borne in mind.

In the domain of algebra a feeling for the situation is less prevalent. The students find it harder to follow the teacher's explanations than they do in geometry because they have less adequate notions of the things involved. Hammering home the notion that algebra is just an expanded arithmetic and that all its formalism is a brief way of stating something which in ordinary words would be cumbersome (the occasional practice of translating the relations of an algebraic formula into more ordinary English helps to bring what is going on down from the clouds into

the grasp of the student) enables the pupil to transfer his arithmetical intuitions, if any, to algebra.

An honest use of mathematical intuition is brought out in the search for a solution of a problem. By dint of experience gained by imitating one's way through the mazes of type problems already worked out in full by some one else, the learner comes to have a storehouse in his memory of possible ways and means; in some mysterious manner these become welded into a method of procedure in such fashion that frequently one has more or less certainty that a particular problem will yield to a definite method of attack. There is associated with this particular form of consciousness the power of generalization which enables one to see similarities in things apparently diverse, and now and again we have general methods that we can prove will operate successfully upon all problems of a determined class. More often we are content to realize that our general methods are very powerful, and we do not bother with the exact extent of their use.

It seems to me that one ought not to be content with a mere solution of the problem, but ought to be fully as interested in the way in which the solution was built up, seeing what were the clues that led to the adoption of the successful approach, and finding out what were the significant features of the problem that were characteristic of its class rather than of it itself as an individual. The general mass of ideas cognate to these I have partially expressed I call the motivation of a problem. I consider that a man has much to learn about a topic if he cannot trace its motif. The close scrutiny of the means employed in one problem will rather often come to your aid when you are stuck on the next. Students who have come to have intuitions about the possible ways of doing a problem, intuitions that are clear enough to denote the probable optimal methods, may be said to have acquired a very useful faculty, that of mathematical motivation. As far as that goes, much of this discussion would seem applicable to other fields.

There is a certain habit which, were it but the property of every pupil, would go far towards brightening the life of the teacher, and that is the habit of subjecting all results of inquiry to a rational examination as to their likelihood of being true, or of being reasonable. After having found what he considers the solution of a problem, the ordinary mortal is frequently apt, if he possesses little native interest in the subject, merely to *hope* that his work is correct. He rarely thinks of trying to get any kind of check on his work, least of all does he consciously endeavor to cultivate an attitude of mind which is inherently critical. It is by no means difficult, in the majority of problems which have a numerical answer, to go through a rough and ready two figure computation mentally, which will check the order of magnitude of the answer. Recently one of my freshman classes was computing at the blackboard the time it takes light to travel to us from the nearest fixed star whose parallactic angle was given. Most of them got the answer correct in miles, but in turning over to light years the would-be results ranged form eighteen weeks to twenty years. Not one of them had the ghost of an idea which limit was the more reasonable. Their background was still too sketchy. They had not formed the habit of improving the quality of their snap judgments.

Returning to the original tobacco problem which stands at the head of this paper, we have an example of what I mean by a rational guess. It is by no means exceedingly hard to practice checking up rough guesses, and the constant revision of first impressions finally results in an ability to feel whether or not an answer is a reasonable one. A case in point is that of a young lad I know who added to his natural aptitude for mathematics this practice in trying to make rational guesses. It so happened that he had a chance to enter a contractor's office, and found there that despite his ignorance of the special problems of the trade he could very frequently estimate as closely as the old-timers by subjecting his rough first impressions to this check of rationality. Naturally as the work grew more

familiar, his background grew better, and his judgments more accurate.

Many students show a distinct betterment in daily work after it has been dinned into them that they can frequently weed out mechanical and even logical errors by wondering whether or not their answer seems probable.

Summary.—It may well be that students can improve their mathematical powers by supplementing their formal work with common sense intuitions, paying particular attention to the dominant guiding principles which enter in the solution of their problems, and constantly checking up on their ideas and results in such fashion that they will not care to submit a result which is more or less obviously ridiculous or improbable.

SYMBOLISM IN MATHEMATICS

By Albert A. Bennett, University of Texas

Summary:

I. General: (1) Mathematics counted among the natural sciences; (2) symbols not necessary for thought but needed for growth of science; (3) science to be distinguished as to use of symbols from ritualism on the one hand and from descriptive art on the other; (4) science classifies its symbols.

II. Mathematics in particular: (1) Mathematics almost a pure science of symbols; (2) mathematical existence; (3) mathematics distinguished by type statements; (4) suitable notation makes formulas possible and computation convenient; (5) universal symbolic logic; (5) symbols are economical and aid in generalization; (7) symbols reduce computations to a mechanical art but cannot replace thought and ingenuity in mathematical progress.

The place of symbolism in mathematics can be rightly appreciated only when the question is treated in its larger aspect in relation to science in general. Mathematics is the first of the sciences in the antiquity of its inception, the purity of its logic, and the intricacy of its ramification. Philosophical speculations on the classification of knowledge has led some thinkers not intimately versed in any of the sciences to speak of mathematics as of a thing apart, not itself a science although related to all sciences. Whatever terms one may use, and it is fruitless to dispute over distinctions that others may draw, there seems to be no doubt that there is a close kinship in spirit and point of view between investigators in the more advanced fields of natural science usually so-called and those in mathematics. There is the same development of trained intuition, the same critical testing of surmises, the same ingenious experimental application of general principles, the same clarifying of fundamental concepts.

Thought is possible without verbal or visual symbols. At least it is safe to say that for many dumb animals, a situation meets with a response that had we observed it presented to a human being, we would have called the resulting act the evident effect of thought. There is no real contribution made to the study by defining thought as a purely human phenomenon. Something certainly has occurred and a new name for the mind process merely provides a distinction without a valid difference. Not only is this wordless reasoning possible, but ideas or at least information can be transmitted without language in the proper sense. Evidence is furnished by widely differing animal communities even down to the insects, abundantly confirming this statement. Nevertheless such a thought cycle as receiving impressions, forming ideas, reasoning upon data secured, transmitting conclusions to others, is immeasurably facilitated by the use of language. And language is but symbols and their orderly use. Were there but a single extant language, one might imagine perhaps that the word used as a symbol had also some intrinsic relation to the object denoted, but the multiplicity of words in any one language and even more the diversity of tongues reminds us constantly of the purely symbolic character of speech and this applies even when a word shows traces of onomatopoeia.

Mathematics, like every other science, employs language, and so indulges continuously in symbolism. But science may be distinguished in its use of symbolism from two other types of human expression, namely, ritualism and descriptive art. By ritualism, I mean here the use of traditional symbols whose meaning has been largely lost to the lay mind. The essence of a magical formula is that the tokens themselves may be trivial or meaningless but the whole incantation has an unreasoned potency. The mysteries of priestly practice are intended to be felt and not analysed by the layman. They must be such as to suggest mysterious origins transmitted by immutable traditions from antiquity. The symbols must be seen or heard and

recognized but their interpretation must be among the arcana of hierarchy hidden perhaps even from the high priest himself. In descriptive art, on the other hand, as I shall use the term, the individual symbols are noted and analytically appreciated chiefly by the artist. The aim is to convey ideas not explicitly enunciated, to suggest more than is expressed, to hint at depths, at brilliance, at grandeur, or grace or simpilicty that is not present save by implication from the total composition. The boundlessness of the ideas to be conveyed, and the inherent limitations of the medium employed determine the scope for the true artist whether in drama, architecture, painting, sculpture, or music. Ritualism and descriptive art are like science governed by certain general esthetic principles in so far as they correspond to expressive forms of mental activity; they all show symmetry, proportion, and so forth. But science takes a middle course between the sub-significance of ritualism and the super-significance of descriptive art.

In science the symbol must accurately correspond to the thing symbolized. It must have no traditional and occult connotation on the one hand nor must it hint at imaginary wealth of meaning on the other. Religion may teach the universal potency of some sacred symbol, and even a prosaic ejaculation like "amen" may take on a special odor of sanctity. Science must discard or modify the sense of any ancient concept that no longer fits into the texture of our present-day theories of the universe. The astronomer no longer speaks of the seven crystal spheres of the heavens, and though he talks of planets, he does not intend to suggest aimless wandering. The physicist no longer refers to the heat substance, phlogiston, and when he speaks of atoms he knows that they are not inherently indecomposable. The zoölogist no longer discusses the distribution of dragons, basilisks, fauns and centaurs, but when he talks of the innominate bone he knows that he has named it. But it is not sufficient that the scientist rejects obsolete terms or modifies them to correspond to modern theory.

He must equally guard his language from unjustified inferences, from deductions that are not intended. One of the simplest conditions needed for securing this is to observe the precept, one name for one thing. Imaginative literature revels in a rich vocabulary. Fine shades of meaning may be expressed and an attitude of mind engendered that transcends direct expression. Precisely the same literal thought may be invested in the sonorous cadence of classical phraseology, or hewed out in rough bare outlines in Anglo-Saxon words, and the reader or hearer feels at once a difference that he may be at a loss to account for. Fashions and modes are apparent in the historical trend of scientific research but the body of science cannot vary with whime. It is international in its claims and nontemporal in its principles. Though a single bug, or bird or blossom may have in a hundred digerent localities more than as many names, the technical label of the species is except in isolated gaps of ignorant confusion, unique. One cannot be surprised that childhood names for familiar objects should under other circumstances come to be applied by simple analogy to somewhat similar things, particularly by those whose language is influenced by immediate practical contingencies only. It is perhaps inevitable that "buttercup" should mean any yellow flower of a certain general size and shape, that the calla should be called a lily, and flycatchers be known as hawks, or that asparagus should be widely corrupted to "sparrow-grass," while such facts as that the pear is a rose, the onion is a lily, the spider, is an emancipated shrimp, the squirrel little more than a tree-climbing rat, and the earth only a planet such as Mars, should be passed over without realization or interest by most people.

Science classes its symbols. Not only must each name correspond to a fairly well-defined concept—significant distinctions leading to the introduction of new terms,—but in the classification of knowledge that raises learning to a science these terms become necessarily classified. The introduction of a system of nomenclature at once elastic and

comprehensive is one of the first steps in the establishment of any science. But the system must be more than these. It must reflect in its own structure the relations within the universe of data with which the science deals. The scientist whether he be mathematician, biologist, chemist, physicist, or what not, faces continuously the slowly developing problem of a suitable system of notation.

But let us examine more intimately those questions which relate to mathematics and which identify its distinguishing attributes, and drop all further discussion of the common features that scientific inquiry universally displays.

Mathematics comes near to being a pure science of symbols. While symbols are essential in the practical discussion of any science, there is yet an objective reality to physical data which suggests that names are largely a sort of short-hand and the things themselves could be displayed and studied without the use of distinguishing notation. In mathematics the very objects under investigation are usually symbols, of quantity, of operation, or of relation. In a fair sense of the terms one could say that to abolish symbols would be to abolish mathematics.

Mathematical existence in nothing other than logical existence and this differs in many ways from physical existence. While in the physical world it may be true in a certain philosophical sense that whatever could now be existent does now exist, the explicit inconsistency in supposing existent something we believe not to exist might be extremely difficult to establish and would be in many instances a hopeless and unheard of problem. Yet in the realm of mathematical concepts whatever is not inconsistent must exist, and the usual exercise is to try to prove directly the inconsistency or else exhibit the assumed object. In the world of mathematics, things last for ever. They cannot die out, become extinct and wholly perish from the universe, nor can they be generated as a new species. They may be newly discovered, or it may prove, as in the case of dragons, that after all they never existed, although in our

ignorance we may have talked about them. Without symbols the very objects of mathematical thought could not be captured or identified. There have been discoveries in chemistry, astronomy and biology of the startling sort where the scientist commences by arguing that if there be an object which has such and such characterisics it must also have such and such other attributes, and, furthermore, it must be found in such and such a place and—behold, there it is! The location of Neptune from the observed perturbations of the outer known planets, the identification of the elements missing from Mendelejev's initial table, the fruitful search for "missing links" among fossil remains, and other physical demonstrations of the existence of things at first only shown to fill certain logical lacunae in the assumed structure of the material world, are among the most sensational feats of natural science. In the realm of pure thought analogous discoveries are in the nature of things frequently trivial. One of the most familiar types of mathematical exercise is that of what expression will satisfy certain conditions, the existence of the expression being established if the expression is found by the very fact of its discovery. In the more difficult cases one attempts either to prove the conditions inconsistent or else to demonstrate the existence and determine the properties of the expression postulated.

But although mathematics deals with symbols for its subject matter, this simple fact alone would not distinguish its procedure from that of any other philosophical inquiry whether with regard to theology, esthetics, politics, or any other even more distantly removed field. The scientific character of mathematics makes symbolism appropriate for the statement of its propositions, while symbolic sentence forms would be almost fatal to some of those other lines of thought. The use of type sentences as rigid unyielding molds into which various terms may be cast, leads to sharply defined and familiarly recognized forms, to the general benefit of the science, but to the havoc of mystic speculation. The philosopher may revel in an unceasing variety,

an endless row of figures of speech, of illustrations, literary allusions, and rhetorical appeals to convey and emphasize the vague or complicated notions he may then be trying to express, but these are the very temptations that the mathematician seeks to avoid. We may illustrate the application of symbolism to sentence form by the following mathematical expressions:

1. The line determined as passing through the point A, also through the point B, is in the same plane as the line determined as passing through the point C and also through the point D, and these lines will not meet however far both may be produced.
2. Line AB is parallel to line SD.
3. AB || CD.

One may note here successive stages in the formal condensation of sentence symbols. And yet this example taken as it is from the subject of geometry, a subject hampered with informal intuition and complex axioms cannot serve as naturally for an illustration as almost any formulation of mathematical analysis such as is furnished by algebra, calculus, differential equations, or other such familiar branches. The extensive development of algebra without such symbols as are now current or other equally concise and equally suggestive ones is difficult to conceive of. Widespread comprehension of commercial arithmetic is almost dependent upon the historically recent introduction of a symbol for zero, and of the use of place values for digits, such as that employed in the Arabic system of notation. If any reader is inclined to doubt this statement, let me suggest that he try his hand at adding the following fractions, all computations being performed with Roman numerals: (one need hardly remark that the following notation is strictly mongrel) (XLIV/XCVII+CCCXLVIII/CDXCIX), or solve without rewriting the following elementary quadratic equation: Twelve times the expression obtained by multiplying the quantity: the unknown increased by one, by itself, all of this diminished by four times the number which exceeds the unknown by three, the entire

expression so far given being augmented by thirteen is given as equal to seven times what is obtained by diminishing the unknown by five, the second member so far stated being increased by eleven units. In short, symbols constitute not only the set of objects with which mathematics is concerned, but very fortunately provide most of the language with which a considerable part of the current mathematical study is carried on.

Modern computation uses symbols almost exclusively. In one sense this is inevitable since symbols must be used for the numbers themselves. But it is of interest to note that most computation is nonverbal. Despite certain oral arithmetic processes, such as illustrated in the successive steps of the soliloquy, "two plus three is five, and seven is twelve, and five is seventeen, and four is twenty-one," and as is heard in such phrases as "carry the two," when adding columns of figures, such physically existent representations as the fingers on the two hands, counters on an abacus, marks on parchment, or the like, have been used as aids in computation since prehistoric times, and without need for other words or signs. In the statement of proofs, however, there has been little changes in the centuries since Euclid, where verbal phrases and sentence play the chief role. A German book on mathematics will contain German words and German grammatical constructions, and no one would confuse it with a French work on the same topic, although, of course, such elementary exercises as "$7x+9=23$, $x=?$" might appear in the same form in both books.

Many conspicuous mathematicians today are themselves using and are urging upon others the use of a universal mathematical, or more properly, logical language of symbols, not like Esperanto, but a compact, stereotyped language with few logical connective signs, in terms of which the classical theorems of analysis can be completely expressed without the use of words. Such a language takes time—a little time—to learn to read and write, although

part of this time is devoted to the mere grasp of certain formal concepts. The language is already a medium of publication in Italy, Great Britain, and the United States and may before long entirely replace in works of mathematical research the psuedo-scientific provincialism that shows itself in the persistence of literary phrases. One of the chief motives for its use lies not in the international character of the work published in it, but rather in the brevity and clarity which it demands and exhibits. The strict use of mathematical and logical symbolism makes possible and convenient the study of complicated propositions whose statements in the current fashion would be liable to ambiguity and whose comprehension would be a severe strain upon the attention and ingenuity of the reader.

The use of symbols in connection with mathematical operations is essentially economical. It economizes in time for reading and for writing, in space for publication, but primarily it saves effort. It renders visually obvious what might take much mental attention and concentration to grasp and largely eliminates the waste of misinterpretation. But the use of symbols does more. It not only introduces economy into mathematics just as it does for the structures and reactions of organic chemistry, but it serves to make generalizations convenient and obvious. Much as the use of a single variable, x, in algebra makes it possible to cover an infinite number of numerical cases by means of the equations developed for the handling of a single case, so the study of a propositional function with a symbol for an unknown relation may frequently be observed to be of immediate applicability in cases never dreamed of at the start.

Symbols cannot render the thinking process obsolete in mathematics, but they have made formulas possible. Many problems which at one time baffled the ingenuity of experts have been reduced by the systematic use of symbols and formulas to mere exercises in computation. What once demanded thought has in such a case become wholly mechanical. The field of computational problems is being gradually enlarged. The outstanding landmarks of any

given field are being slowly incorporated within the orderly realm of well mastered topics as the result of systematic and intelligent effort. But one must not suppose that this implies that the domain of mathematics is being exhausted and that the room for thought is becoming cramped. Precisely the contrary is the case. Each newly won outpost reveals wider stretches of untraversed territory, new scope for high intelligence, trained intuition, and well digested knowledge. Symbols have never before played so inconspicuous part in the growth of new discoveries, nor, one may add, so large part in the exposition of results. By the critical use of symbols mathematics has freed itself of much that was accidental and extraneous, and is less bound than ever before by formalism and computational conventions. Thought works with and through symbols, but no mere juggling of signs, no mere computational drill, no mere chaining together of propositions, can ever usurp the glory of mathematical discovery, impelled by a pure thirst for truth, guided by the intuition of genius and established by the rigorous exactions of pure logic.

FRESHMAN PRIZE EXAMINATION

The annual Brown University Mathematical Prize examination for Freshmen was held at the University of Texas, Saturday, October 25, 1924, and lasted for one hour, from 1:45 to 2:45 P.M. . . The following were the questions proposed:

1. Solve

$$2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} - \frac{6a}{b}$$

2. If four positive quantities are in continued proportion, $(a:b = b:c = c:d)$, show that numerically the difference between the first and last is at least three times the difference between the middle two.

3. From a given point on a given circle draw a chord bisected by a given chord.

4. Find the locus of a point such that the difference of the squares of its distances from two given points is a given constant.

The prize winners are as follows:

First prize, Alvin Kilgore, Victoria High School.

Second prize , Louis Statham, Brackenridge High School, San Antonio.

Third prize, John C. Cramer, Central High School, Houston .